

Symmetries of the near horizon of a Black Hole by Group Theoretic methods

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Abstract

We use group theoretic methods to obtain the extended Lie point symmetries of the quantum dynamics of a scalar particle probing the near horizon structure of a black hole. Symmetries of the classical equations of motion for a charged particle in the field of an inverse square potential and a monopole, in the presence of certain model magnetic fields and potentials are also studied. Our analysis gives the generators and Lie algebras generating the inherent symmetries.

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1 Introduction

In certain physical problems there may exist extra hidden symmetries which are not apparent, unless searched for. Some of the examples from classical considerations are, a particle in a $\frac{1}{r^2}$ potential in one dimension [1] and its quantum version in any dimension[3], two dimensional delta function potential in spatial two dimensions[4], the conserved Runge-Lenz vector of the Kepler problem and the extra symmetries of a charge moving in the field of a magnetic monopole[5, 6], and the generators beyond the Poincare invariance that give rise to conformal invariance in electrodynamics as well as in Yang-Mills theory. The existence of symmetries help in classifying and obtaining energy levels and eigenstates in quantum mechanical problems, generating new solutions and also formulating conservation laws. As we know, manifestation of scale invariance in deep inelastic scattering had deep significance in the development of gauge theories. The invariance under scale and conformal transformations also motivated the construction of a simple classical model which leads to conformal quantum mechanics[1]. Recently there have been a revival of interest in this model. This is due to the observation in string theory dynamics that a particle near a black hole possesses $SO(2,1)$ symmetries as in conformal quantum mechanics[2].

The symmetries of charged particle - monopole system and the conformal quantum mechanics were obtained from physical reasonings and scale invariance in [1, 5]. However, there exists a general program to obtain the symmetries of the equations of motion of any such system by using the group theoretic methods of Lie. In this paper we use this method to find the Lie point symmetries of the inverse squared potential and the monopole system as well as some other physically motivated systems.

A knowledge of the symmetry group of a system of differential equations leads to several types of applications[10, 25, 26] such as finding the solutions, constructing new solutions from the known ones . Further, just the enumeration of the symmetry generators sometimes provide much physical insight and quantitative physical results for which the full solutions are not required. The derivation of Kepler's third law of planetary motion and Runge-Lenz vectors[17], calculation of energy levels for hydrogen like atoms and generalized Kepler's problems, harmonic oscillators, Morse potentials[18], electron in a specific nonuniform magnetic field[19], being some such examples. In these the energy eigenvalues are obtained through the method of spectrum generating algebras which gives the Casimir invariants directly without explicit recourse to the solutions. Of course, the solutions are also obtained from the representation theory. For finding the continuous symmetries, Lie's

method of group analysis seems to be the most powerful technique available. For example, Witten[20] had considered an example of the equation of motion of a particle in three dimensions constrained to move on the surface of a sphere in the presence of a magnetic monopole. This is the classical analogue of the Wess-Zumino model[21]. The equations of motion of such a system in the presence of a magnetic monopole cannot be obtained from the usual Lagrangian formulation unless one goes to a higher dimension. Hence, the usual method of finding the symmetries through Noether's theorem would have difficulties. So, to look for the continuous symmetries associated with such classical systems one has to analyze directly the equations of motion. Similar is the case for Korteweg-de Vries equation which is not amenable to a direct Lagrangian formulation when expressed as a lowest order equation[22]. Another example is the Lorenz system of equations which have been dealt in the papers by Sen and Tabor[23], and Nucci[16]. For the classical systems, this procedure of finding directly the symmetries from equation of motion is, in some sense, more fundamental. This is because in certain cases many different Lagrangians may give rise to the same equations of motion. A group analysis of the equations of motion gives all the Lie point generators of the symmetry group. In the cases where a Lagrangian formulation is possible, the usual Noether symmetries are a subset of the above generators. This subset of generators acting on the Lagrangian gives zero[17]. However, besides these there may be other generators obtained through group analysis which have direct physical significance, but not explicitly available from the consideration of usual Noether symmetries alone[24, 11, 12, 15]. The reproduction of Kepler's third law in the planetary motion problem is such an example. The extension of this idea to the notion of Lie dynamical symmetries contains similarly a subclass known as Cartan symmetries. The Runge-Lenz vector can be obtained from such considerations. These symmetries are, further, related to the Lie-Bäcklund symmetries.

The application of these types of analysis to nonlocal cases have been widely studied through Bäcklund transformations and related techniques in the context of integrable systems containing infinite number of conservation laws[27]. In [13] a hereditary recursion operator for the Harry-Dym equation generating infinitely many Lie-Bäcklund symmetries have been found. The nonabelian prolongation structure of the cylindrical Korteweg-de Vries has been used to derive a set of Bäcklund transformations and a nonlinear superposition formula in [14]. The Thirring model has been analyzed by Morris [28, 29]. The differential geometric forms developed earlier are used in the above analysis to obtain the prolongation structure[30]. Some other appli-

cations of these ideas to important problems from physics is comprehensibly covered by Gaeta[31].

In this paper, we find the Lie point symmetries of equations representing the motion of a charged particle in three dimensions under the influence of (i) a $\frac{1}{r^2}$ potential, (ii) in field of a monopole, (iii) classical Wess-Zumino-Witten model, (iv) a dyon, and, (v) a scalar field probing the near horizon structure of a black hole.

2 Symmetry generators, classical particle

Typically we are interested in the coupled set of equations representing the equations of motion of a particle in three dimensions. These are of the form

$$\ddot{x}_a = \beta \omega_a(x_i, \dot{x}_i, t) \quad (1)$$

where a dot represents derivative with respect to time, $a, i = 1, 2$, and 3 , and β is a constant involving mass, coupling constant etc. which we set equal to one. Following Stephani [17], we will find the infinitesimal generators of the symmetry under which the system of differential equations does not change. The symmetry is generated by \mathbf{X} and its extension

$$\dot{\mathbf{X}} = \tau \frac{\partial}{\partial t} + \eta_a \frac{\partial}{\partial x_a} + \dot{\eta}_a \frac{\partial}{\partial \dot{x}_a} \quad (2)$$

and the symmetry condition under transformations represented by the following equation determines τ and η_a s. In the expanded form the symmetry conditions are given by

$$\begin{aligned} & \eta_b \omega_{a,b} + (\eta_{b,t} + \dot{x}_c \eta_{b,c} - \dot{x}_b \tau_{,t} - \dot{x}_b \dot{x}_c \tau_{,c}) \frac{\partial \omega_a}{\partial \dot{x}_b} \\ & + \tau \omega_{a,t} + 2\omega_a(\tau_{,t} + \dot{x}_b \tau_{,b}) + \omega_b(\dot{x}_a \tau_{,b} - \eta_{a,b}) \\ & + \dot{x}_a \dot{x}_b \dot{x}_c \tau_{,bc} + \dot{x}_a \tau_{,tt} + 2\dot{x}_a \dot{x}_c \tau_{,tc} \\ & - \dot{x}_c \dot{x}_b \eta_{a,bc} - 2\dot{x}_b \eta_{a,tb} - \eta_{a,tt} = 0 \end{aligned} \quad (3)$$

where $f_{,t} = \frac{\partial f}{\partial t}$ and $f_{,c} = \frac{\partial f}{\partial x_c}$.

The solutions of the symmetry conditions provide us the generators of the group.

In the presence of a potential like $\frac{1}{r^2}$ we find the generators with extensions to be

$$\dot{\mathbf{X}}_a = \varepsilon_{abc} \left(x_c \frac{\partial}{\partial x_b} + \dot{x}_c \frac{\partial}{\partial \dot{x}_b} \right), \quad \text{space rotations,}$$

$$\begin{aligned}
\dot{\mathbf{X}}_4 &= \frac{\partial}{\partial t}, \quad \text{time translation,} \\
\dot{\mathbf{X}}_5 &= 2t \frac{\partial}{\partial t} + x_a \frac{\partial}{\partial x_a} - \frac{1}{2} \dot{x}_a \frac{\partial}{\partial \dot{x}_a}, \\
\text{Kepler like scaling law } \frac{t}{r^2} &= \text{constant,} \\
\dot{\mathbf{X}}_6 &= t^2 \frac{\partial}{\partial t} + tx_a \frac{\partial}{\partial x_a} + x_a \frac{\partial}{\partial \dot{x}_a} - \dot{x}_a \frac{\partial}{\partial \dot{x}_a}
\end{aligned} \tag{4}$$

The vector fields have the commutation relations

$$\begin{aligned}
[\mathbf{X}_a, \mathbf{X}_b] &= \varepsilon_{abc} \mathbf{X}_c, \\
[\mathbf{X}_a, \mathbf{X}_4] &= [\mathbf{X}_a, \mathbf{X}_5] = [\mathbf{X}_a, \mathbf{X}_6] = 0 \\
[\mathbf{X}_4, \mathbf{X}_5] &= 2\mathbf{X}_4, \quad [\mathbf{X}_4, \mathbf{X}_6] = \mathbf{X}_5, \quad [\mathbf{X}_5, \mathbf{X}_6] = 2\mathbf{X}_6
\end{aligned} \tag{5}$$

The classical Kepler problem with $\frac{1}{r}$ potential has a different scaling law of $\frac{t^2}{r^3}$ and also does not possess the symmetry corresponding to generator \mathbf{X}_6 . However, it possesses a Runge-Lenz vector. Stephani has given a general method to obtain such conserved vectors in the Lagrangian formulation. In the quantum mechanical case, if the eigenvalues are taken instead the Hamiltonian operator, an enhanced symmetry occurs for $\frac{1}{r}$ potential. For $\frac{1}{r^2}$ potential we could not find a classical Runge-Lenz vector by Stephani's method. This appears to be related to orbits being not closed in such a potential[33, 34, 35]. However, as has already been noted, in this case new vector fields result leading to the extra symmetries.

For the quantum mechanical case with a $\frac{1}{r^2}$ potential in the Schrödinger equation, Jackiw has given the physical argument that in any dimensions the kinetic term scales as $\frac{1}{r^2}$ and an $SO(2,1)$ symmetry results[3]. Jackiw has also considered the symmetries of equation of motion, Lagrangian, and Hamiltonian for a charged particle in the field of a magnetic monopole [5, 7, 8]. He had discovered an extra $SO(2,1)$ hidden symmetry by scaling and physical considerations. Leonhardt and Piwnicki have explored the theoretical possibility of obtaining the field of quantized monopoles when a classical dielectric moves in a charged capacitor [36]. Since the magnetic field due to a magnetic monopole is $B_a = \frac{x_a}{r^3}$, the equation of motion is

$$\ddot{x}_a = \varepsilon_{abc} \frac{\dot{x}_b x_c}{r^3} = \omega_a \tag{6}$$

The Lie symmetries of this equation were obtained in[38, 39] and are identical to those of the $\frac{1}{r^2}$ potential found by us. Such a potential was added by Zwanzinger in his analysis of monopole system[37].

3 Symmetry generators, quantum particle

The simplest one dimensional version of the equation

$$\ddot{x}_a = \frac{\mu^2 x_a}{mr^4} \quad (7)$$

possesses remarkable symmetries which were exploited by de Alfaro, Fubini, and Furlan to construct conformal quantum mechanics. Here x is considered as a field in zero space and one time dimension. The quantum mechanical equation for the wave function u becomes[1], in our notation,

$$\left(-\frac{d^2}{dr^2} + \frac{g}{r^2} + \frac{r^2}{a^2} \right) u = \frac{4b}{a} u \quad (8)$$

where $\frac{\mu^2}{m}$ is replaced by g . Here a is a constant which plays a fundamental role in the theory and b is related to appropriate raising and lowering operators. This equation can be expressed in terms of differential operator realization of $su(1,1)$ algebra[18] and was studied in detail in[1].

There have been earlier works, where it has been shown that the dynamics of a scalar particle approaching the event horizon of a black hole is governed by an Hamiltonian with an inverse square potential[42, 43, 44, 45, 46, 47, 2]. The scalar field can be used as a probe to study the geometry in the vicinity of the horizon and its dynamics is expected to provide clues to the inherent symmetry properties of the system. The Hamiltonian of conformal quantum mechanics fits into this. This Hamiltonian also arises as a limiting case of the brick-wall model describing the low energy quantum dynamics of a field in the background of a massive Schwarzschild black hole of mass M [46, 47]. On factorizing such a Hamiltonian a Virasoro symmetry was found by Birmingham, Gupta and Sen. They have studied the representation of the algebra as well as the scaling properties of the time independent modes[2]. The full Virasoro algebra was obtained by the requirement of unitarity of the representation. The Hamiltonian operator is in the enveloping algebra.

However, here we aim at finding the underlying Lie point symmetry of the equation of motion of the scalar particle, viewed as a differential equation. Of course, mathematically any two linear homogeneous ordinary differential equations can be transformed to the form, where a prime denotes a differentiation with respect to r ,

$$u'' = 0. \quad (9)$$

This equation has the eight dimensional symmetry of projective transformations. But the two equations could be quite different from physics point of view having different eigenvalues and eigenfunctions. Hence we would like to see explicitly what are the Lie point symmetries of the particular equation.

For the equation

$$u'' = \omega(r, u, u') \quad (10)$$

where

$$\omega(r, u, u') = -\left(\frac{C}{r^2} + \frac{D}{r} + \hat{E}\right)u(r). \quad (11)$$

the symmetry generators are obtained from the conditions given by the equation (3) which reduces in the one dimensional case to

$$\begin{aligned} & \omega(\eta_{,u} - 2\tau_{,r} - 3u'\tau_{,u}) - \omega_{,r}\tau - \omega_{,u}\eta \\ & - \omega_{,u'}[\eta_{,r} + u'(\eta_{,u} - \tau_{,r}) - u'^2\tau_{,u}] + \eta_{,rr} \\ & + u'(2\eta_{,ru} - \tau_{,rr}) + u'^2(\eta_{,uu} - 2\tau_{,ru}) - u'^3\tau_{,uu} = 0 \end{aligned} \quad (12)$$

For the case $C = -g$, $D = \hat{E} = 0$, equating to zero the coefficients of u'^3 and u'^2 in (12) we get

$$\tau_{,uu} = 0, \quad \eta_{,uu} = 2\tau_{,ru} \quad (13)$$

which are satisfied for

$$\tau = u\alpha(r) + \beta(r), \quad \eta = u^2\alpha'(r) + u\gamma(r) + \sigma(r). \quad (14)$$

Using these and equating to zero the coefficient of u' and then considering the the terms not involving u' , we find that an interesting symmetry exists only when the coupling constant g is equal to 2. For this case we obtain

$$\tau = \frac{1}{r}Au + Fr, \quad \eta = -\frac{1}{r^2}Au^2 + Bu + \sigma(r) \quad (15)$$

where A, F , and B are constants and $\sigma(r)$ satisfies the same equation as u does. The vector fields are

$$\mathbf{X}_1 = r\frac{\partial}{\partial r}, \quad \mathbf{X}_2 = u\frac{\partial}{\partial u}, \quad \mathbf{X}_3 = \frac{1}{r}u\frac{\partial}{\partial r} - \frac{1}{r^2}u^2\frac{\partial}{\partial u} \quad (16)$$

with commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \quad [\mathbf{X}_1, \mathbf{X}_3] = -2\mathbf{X}_3, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3 \quad (17)$$

For the case, considered in [9], the relevant equation is

$$\frac{d^2 u}{dr^2} + \frac{1}{r^2} \left[\frac{1}{4} + R^2 E^2 \right] u = 0 \quad (18)$$

where E is a generic eigenvalue and $R = 2M$. Hence g corresponds to $\left[\frac{1}{4} + R^2 E^2 \right]$ in this case. This equation corresponds to the Hamiltonian of the form

$$H = \frac{p^2}{2} + \frac{g'}{2r^2} \quad (19)$$

where $-\frac{g'}{2} = \left[\frac{1}{4} + R^2 E^2 \right]$. Along with H , the generators of dilations

$$D = \frac{(pr + rp)}{2}, \quad (20)$$

and special conformal transformations

$$K = \frac{r^2}{2} \quad (21)$$

form the $SL(2, R)$ conformal algebra. But since the D and K do not commute with the Hamiltonian, as is well known, $SL(2, R)$ is not a symmetry of the theory[48]. However, $SL(2, R)$ can be used to relate states of different energies. Our analysis of the eigenvalue equation shows, in contradistinction to the above formulation, what the symmetries are. However, only when E is imaginary with $\left[\frac{1}{4} + R^2 E^2 \right] = -2$, the symmetry will show up. This reminds of the situation in spontaneous symmetry breaking in gauge theories where the mass squared value has to become negative. But in our case, on the other hand, the symmetry pops up at the particular imaginary E . The one parameter groups G_i generated by the vector fields \mathbf{X}_i are given by,

$$\begin{aligned} G_1 &: (e^\epsilon r, u), \\ G_2 &: (r, e^\epsilon u), \\ G_3 &: \left(\left(1 + \frac{\epsilon u}{r} - \frac{\epsilon^2 u^2}{r^4} \right) r, \frac{\exp\left\{-\frac{\epsilon u}{r^2}\right\}}{\left(1 - \frac{\epsilon^2 u^2}{r^2} \right)} u \right). \end{aligned} \quad (22)$$

up to order ϵ^2 for G_3 . The wave function exhibits scaling behaviour at this imaginary E for the generators \mathbf{X}_2 and shows a much complicated symmetry behaviour corresponding to nonlinear transformation involving u and r for the vectorfield \mathbf{X}_3 .

We note here that the $\frac{1}{r}$ and $\frac{1}{r^2}$ factors in \mathbf{X}_3 makes it ill defined as $r \rightarrow 0$ similar to the L_{-n} operators of conformal field theory or the P_m operators considered by Birmingham, Gupta, and Sen[2].

4 Conclusion

The equations of motion of a free particle, admit eight symmetries for each of the x_a s. This is the maximum number of symmetries for an ordinary second order differential equation. By including different x_a , \dot{x}_a dependent terms in the equations we do explicitly see which generators survive as symmetries and we have found corresponding complete Lie algebras. We choose some cases motivated by problems from physics. The original motivation of including Wess-Zumino terms in the Lagrangian has been to reduce some of its symmetries[32], By above type of analysis we find that the equations of motion now support the three dimensional rotations and a time translation symmetry instead of the six vector fields as for the monopole problem without any constraint. A calculation gives the same four symmetries for dyon. It should be noted that in the context of the symmetries of Wess-Zumino-Witten models, the symmetries in the higher dimensions play by far the most important role and these have been fruitfully exploited [40, 41].

The Lie symmetries correspond to the transformations of the solutions. An analysis for Dirac equation modified by nonlinearity has been carried out in [53]. It is also expected that related group analysis may provide useful information when terms are modified in the Lagrangian, due to quantum corrections, for example.

For the case of a scalar particle probing the near horizon structure of a black hole, under certain limits the Hamiltonian contains $\frac{1}{r}$ and $\frac{1}{r^2}$ potentials. Here we find that only for a specific value of the coefficient of $\frac{1}{r^2}$ term and in absence of $\frac{1}{r}$ term there exist symmetry with three generators. The symmetries are a scaling transformation of r , a scaling transformation of the wave function u , and a nonlinear transformation involving both u and r for u . The specific value of energy corresponds to energy squared being negative, whereas for spontaneous symmetry breaking to occur in gauge theories, the mass squared is taken to be negative. We note that in contrast to spontaneous breaking of symmetry, our analysis shows that the symmetry of the eigenvalue equation is enhanced for a particular imaginary value of the energy. We expect that these considerations will ultimately lead to a better understanding of the spontaneous symmetry breaking, as well.

The similar symmetries of an inverse square potential and the monopole system in the classical case is intriguing. Symmetry analysis of the monopole system for the quantum case would lead to a deeper understanding. The quantum mechanical problem of a charged particle in the presence of even a constant magnetic field has many interesting mathematical structures [49, 50] and under certain limits can make space coordinates noncommuta-

tive [54]. Klishevich and Plyuschay have found a universal algebraic structure at the quantum level for the two dimensional case in the presence of certain magnetic fields[49, 50]. Nonlinear superconformal symmetry of the fermion-monopole system has been extensively studied in[51, 52]. It would be worthwhile to analyze the existence of such structures in combination with various r dependent potentials.

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